This result, called a **law of large numbers**, is primarily of theoretical interest. Of much more practical value is the **central limit theorem**, one of the most important theorems of statistics, which concerns the limiting distribution of the **standardized mean** of n random variables when $n \to \infty$. We shall prove this theorem here only for the case where the n random variables are a random sample from a population whose moment-generating function exists. More general conditions under which the theorem holds are given in Exercises 8.7 and 8.9, and the most general conditions under which it holds are referred to at the end of this chapter.

THEOREM 8.3. CENTRAL LIMIT THEOREM. If $X_1, X_2, \ldots, X_n$ constitute a random sample from an infinite population with the mean μ, the variance σ^2, and the moment-generating function $M_X(t)$, then the limiting distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

as $n \to \infty$ is the standard normal distribution.

Proof First using the third part of Theorem 4.10 on page 128 and then the second, we get

$$M_Z(t) = M_{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}(t) = e^{-\sqrt{n}\,\mu t/\sigma} \cdot M_{\overline{X}}\left(\frac{\sqrt{n}t}{\sigma}\right)$$
$$= e^{-\sqrt{n}\,\mu t/\sigma} \cdot M_{n\overline{X}}\left(\frac{t}{\sigma\sqrt{n}}\right)$$

Since $n\overline{X} = X_1 + X_2 + \cdots + X_n$, it follows from Theorem 7.3 that

$$M_Z(t) = e^{-\sqrt{n}\,\mu t/\sigma} \cdot \left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

and hence that

$$\ln M_Z(t) = -\frac{\sqrt{n}\,\mu t}{\sigma} + n \cdot \ln M_X\left(\frac{t}{\sigma\sqrt{n}}\right)$$

Expanding $M_X\left(\frac{t}{\sigma\sqrt{n}}\right)$ as a power series in t, we obtain

$$\ln M_Z(t) = -\frac{\sqrt{n}\,\mu t}{\sigma} + n \cdot \ln\left[1 + \mu_1'\frac{t}{\sigma\sqrt{n}} + \mu_2'\frac{t^2}{2\sigma^2 n} + \mu_3'\frac{t^3}{6\sigma^3 n\sqrt{n}} + \cdots\right]$$

where μ_1', μ_2', and μ_3' are the moments about the origin of the population distribution, that is, those of the original random variables X_i.

If n is sufficiently large, we can use the expansion of $\ln(1+x)$ as a power series in x (as on page 191), getting

$$\ln M_Z(t) = -\frac{\sqrt{n}\,\mu t}{\sigma} + n\left\{\left[\mu_1'\frac{t}{\sigma\sqrt{n}} + \mu_2'\frac{t^2}{2\sigma^2 n} + \mu_3'\frac{t^3}{6\sigma^3 n\sqrt{n}} + \cdots\right]\right.$$

$$-\frac{1}{2}\left[\mu_1'\frac{t}{\sigma\sqrt{n}} + \mu_2'\frac{t^2}{2\sigma^2 n} + \mu_3'\frac{t^3}{6\sigma^3 n\sqrt{n}} + \cdots\right]^2$$

$$\left.+\frac{1}{3}\left[\mu_1'\frac{t}{\sigma\sqrt{n}} + \mu_2'\frac{t^2}{2\sigma^2 n} + \mu_3'\frac{t^3}{6\sigma^3 n\sqrt{n}} + \cdots\right]^3 - \cdots\right\}$$

Then, collecting powers of t, we obtain

$$\ln M_Z(t) = \left(-\frac{\sqrt{n}\,\mu}{\sigma} + \frac{\sqrt{n}\,\mu_1'}{\sigma}\right)t + \left(\frac{\mu_2'}{2\sigma^2} - \frac{\mu_1'^2}{2\sigma^2}\right)t^2$$

$$+\left(\frac{\mu_3'}{6\sigma^3\sqrt{n}} - \frac{\mu_1'\cdot\mu_2'}{2\sigma^3\sqrt{n}} + \frac{\mu_1'^3}{3\sigma^3\sqrt{n}}\right)t^3 + \cdots$$

and since $\mu_1' = \mu$ and $\mu_2' - (\mu_1')^2 = \sigma^2$, this reduces to

$$\ln M_Z(t) = \frac{1}{2}t^2 + \left(\frac{\mu_3'}{6} - \frac{\mu_1'\mu_2'}{2} + \frac{\mu_1'^3}{6}\right)\frac{t^3}{\sigma^3\sqrt{n}} + \cdots$$

Finally, observing that the coefficient of t^3 is a constant times $\frac{1}{\sqrt{n}}$ and in general, for $r \geqq 2$, the coefficient of t^r is a constant times $\frac{1}{\sqrt{n^{r-2}}}$, we get

$$\lim_{n\to\infty} \ln M_Z(t) = \frac{1}{2}t^2$$

and hence

$$\lim_{n\to\infty} M_Z(t) = e^{\frac{1}{2}t^2}$$

since the limit of a logarithm equals the logarithm of the limit (provided these limits exist). Identifying the limiting moment-generating function at which we have arrived as that of the standard normal distribution, we need only the two theorems stated on page 192 to complete the proof of Theorem 8.3. An illustration of this theorem is given in Exercise 8.13 and 8.14.

Sometimes, the central limit theorem is interpreted incorrectly as implying that the distribution of $\overline{X}$ approaches a normal distribution when $n \to \infty$. This is incorrect because $\text{var}(\overline{X}) \to 0$ when $n \to \infty$; on the other hand, the central limit theorem does justify approximating the distribution of $\overline{X}$ with a normal distribution having the mean μ and the variance $\frac{\sigma^2}{n}$ when n is large. In practice, this approximation is used when $n \geqq 30$ regardless of the actual shape of the population sampled. For smaller values of n the approximation is questionable, but see Theorem 8.4.